Ground state of the single qubit Ising model

Define parameters

Define Pauli operators and other

In[51]:=

```
I1 = {{1, 0}, {0, 1}}; MatrixForm[I1]
```

Out[51]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In[52]:=

```
X = {{0, 1}, {1, 0}}; MatrixForm[X]
```

Out[52]//MatrixForm=

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In[53]:=

```
Y = {{0, -I}, {I, 0}}; MatrixForm[Y]
```

Out[53]//MatrixForm=

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

In[54]:=

```
Z = {{1, 0}, {0, -1}}; MatrixForm[Z]
```

Out[54]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In[55]:=

$$H = \frac{1}{\text{Sqrt[2]}} \{\{1, 1\}, \{1, -1\}\}; \text{MatrixForm[H]}$$

Out[55]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Define the Hamiltonian

In[56]:=

```
Hamiltonian = -J Z - h X; MatrixForm[Hamiltonian]
```

Out[56]//MatrixForm=

$$\begin{pmatrix} -J & -h \\ -h & J \end{pmatrix}$$

Exponential of Pauli Matrices

In[57]:=

```
Ex = MatrixExp[I θx X]; MatrixForm[Ex]
```

Out[57]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\theta_x] & i\,\text{Sin}[\theta_x] \\ i\,\text{Sin}[\theta_x] & \text{Cos}[\theta_x] \end{pmatrix}$$

In[58]:=

```
Ey = MatrixExp[I θy Y]; MatrixForm[Ey]
```

Out[58]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\theta_y] & \text{Sin}[\theta_y] \\ -\text{Sin}[\theta_y] & \text{Cos}[\theta_y] \end{pmatrix}$$

In[59]:=

```
Ez = MatrixExp[I θz Z]; MatrixForm[Ez]
```

Out[59]//MatrixForm=

$$\begin{pmatrix} e^{i\,\theta_z} & 0 \\ 0 & e^{-i\,\theta_z} \end{pmatrix}$$

Eigenvector of Hamiltonian

In[60]:=

```
Eigenvalues[Hamiltonian]
```

Out[60]=

$$\left\{-\sqrt{h^2+J^2},\ \sqrt{h^2+J^2}\right\}$$

Compute the eigenvalues and scales by h to have also the case h=0

In[61]:=

```
es = Eigenvectors[Hamiltonian]
```

Out[61]=

$$\left\{\left\{-\frac{-J-\sqrt{h^2+J^2}}{h},\ 1\right\},\ \left\{-\frac{-J+\sqrt{h^2+J^2}}{h},\ 1\right\}\right\}$$

Scale es by h to remove denominator

In[62]:=

```
es = es * h
```

Out[62]=

$$\left\{\left\{J+\sqrt{h^2+J^2},\ h\right\},\ \left\{J-\sqrt{h^2+J^2},\ h\right\}\right\}$$

These eigenvalues are not normalized

Define normalized eigenvalues

In[63]:=

```
normes = Refine[{es[[1]] / Norm[es[[1]]], es[[2]] / Norm[es[[2]]]}, Assumptions → h > 0 && J > 0];
MatrixForm[%]
```

Out[64]//MatrixForm=

$$\begin{pmatrix} \frac{J+\sqrt{h^2+J^2}}{\sqrt{h^2+\left(J+\sqrt{h^2+J^2}\right)^2}} & \frac{h}{\sqrt{h^2+\left(J+\sqrt{h^2+J^2}\right)^2}} \\ \frac{J-\sqrt{h^2+J^2}}{\sqrt{h^2+\left(-J+\sqrt{h^2+J^2}\right)^2}} & \frac{h}{\sqrt{h^2+\left(-J+\sqrt{h^2+J^2}\right)^2}} \end{pmatrix}$$

The rows are the eigenvectors

Out[45]//MatrixForm=

$$\begin{pmatrix} \frac{J+\sqrt{h^2+J^2}}{\sqrt{h^2+\left(J+\sqrt{h^2+J^2}\right)^2}} & \frac{h}{\sqrt{h^2+\left(J+\sqrt{h^2+J^2}\right)^2}} \\ \frac{J-\sqrt{h^2+J^2}}{\sqrt{h^2+\left(-J+\sqrt{h^2+J^2}\right)^2}} & \frac{h}{\sqrt{h^2+\left(-J+\sqrt{h^2+J^2}\right)^2}} \end{pmatrix}$$

Check the eigenvalues for h=0 and J=0

In[65]:=

```
MatrixForm[Refine[Limit[normes, h → 0], Assumptions → J > 0]]
```

Out[65]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

In[66]:=

```
MatrixForm[Refine[Limit[normes, J → 0], Assumptions → h > 0]]
```

Out[66]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$